

Beyond Rainbow-Ladder in bound state equations

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Abstract. In this work we devise a new method to study quark anti-quark interactions beyond simple ladder-exchange that yield massless pions in the chiral limit. The method is based on the requirement to have a representation of the quark-gluon vertex that is explicitly given in terms of quark dressings functions. We outline a general procedure to generate the Bethe-Salpeter kernel for a given vertex representation. Our method allows not only the identification of the mesons' masses but also the extraction of their Bethe-Salpeter wave functions exposing their internal structure. We exemplify our method with vertex models that are of phenomenological interest.

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1 Introduction

Understanding the spectrum of light and heavy hadrons is an important task on our way towards a full understanding of QCD. In order to identify states that can be accounted for as quark-antiquark bound systems and separate them from more complex ones such as tetraquarks, meson molecules or glueballs one needs to develop a framework that makes contact to the details of the underlying quark-gluon interaction. Lattice QCD is one such approach, the functional method using Dyson-Schwinger equations (DSEs) and Bethe-Salpeter equations (BSEs) is another.

In the latter approach, the construction of an approximation scheme that yields an interaction consistent with chiral symmetry and its breaking patterns is a necessary requirement for the description of light mesons. Only then, the Goldstone boson nature of the pseudoscalar bound states are preserved resulting in a massless pion in the chiral limit [1, 2, 3]. This requirement can most easily be met with the rainbow-ladder truncation which has been widely applied for QCD phenomenology [4, 5, 6, 7]. This truncation has, however, limitations. These become visible for excited states [8, 9], states with finite width, or mesons with axial-vector or scalar quantum numbers, where the rainbow-ladder approach does not provide results in agreement with experiment. On a fundamental level, going beyond simple models for the quark-gluon interaction requires a dynamical treatment of the Yang-Mills sector of QCD as well as a treatment of the quark-gluon vertex that includes beyond rainbow-ladder structures [10].

There have been many efforts to go beyond rainbow-ladder. One promising route is to use explicit diagrammatic approximations to the DSE of the quark-gluon vertex [11, 12, 13, 14, 15, 16, 17, 18]. This allows the explicit study

of the effects of the gluon self-interaction [17] as well as pion cloud effects [18] on the spectrum of light mesons. Another promising approach uses explicit representations of selected tensor structures of the quark-gluon vertex [19, 20, 21, 22]. Most of these approaches have in common that they rely on techniques based on the two-particle-irreducible (2PI) representation of the effective action. In this language the interaction kernel is given as the functional derivative of the quark self-energy with respect to the quark propagator as is detailed in Refs. [1, 2].

In this work we use a similar idea. The difference is, though, that instead of employing a diagrammatic representation of the quark-gluon vertex, we use representations of the vertex that depend on the quark propagator explicitly and perform a systematic derivation of the corresponding Bethe-Salpeter kernel. Our approach is similar in spirit but technically different from the one outlined in [20, 21, 22] and therefore serves as a complementary tool. In particular it has the advantage, that not only the mesons' masses but also their Bethe-Salpeter wave-function can be obtained. This opens up the possibility for future studies of structural information such as form factors and distribution amplitudes. In this respect, our approach improves upon the previous ones.

This paper is organized as follows. Section 2 contains basic definitions, central relations as well as the main theoretical ideas. There we explain how we construct ladder and beyond-ladder kernels in general. In Sections 3.1, 3.2 and 3.3 we consider three specific vertex models, derive the corresponding interaction kernels and study the chiral properties of the different constructions. An implementation of these vertex models can be found in section 4 where our numerical results are presented. We conclude in section 5, followed by an appendix with technical details.

2 Theoretical foundation

In this section we discuss the general principles that are at the heart of the techniques used in this work. This will also serve to make some basic definitions and to introduce our notation.

Our starting point is the definition of the quark anti-quark interaction Kernel K as the functional derivative of the quark self-energy Σ with respect to the dressed quark propagator S

$$K_{ab}^{cd}(x, y, z, z') = \frac{\delta \Sigma^{cd}(x, y)}{\delta S^{ab}(z, z')}, \quad (1)$$

where a, b, c and d are Dirac indices and we work in coordinate space. In a similar fashion, the quark self-energy is obtained from the 2PI effective action. The technique given by Eq. (1) is often called 'cutting' since in a graphical language it corresponds to the cutting of a quark line. The 2PI formalism allows for a closed representation of a truncated effective action in terms of a loop expansion [23, 24]. This has the advantage that the validity of symmetries, such as chiral symmetries, can be checked on the level of the effective action. The cutting procedure then generates equations that respect the consequences of the given symmetry. It has to be emphasized, however, that cutting alone is not sufficient. The quark gluon vertex also needs to behave correctly under chiral transformations [2]. An appropriate tool to investigate the transformation properties of the vertex in the momentum space representation is the axial-vector Ward Takahashi identity (AXWTI) which, if fulfilled, guarantees a massless pion in the chiral limit [3]. In the chiral limit this identity reads

$$iP_\mu \Gamma_{5\mu}(P, k) = S^{-1}(k_+) \gamma_5 + \gamma_5 S^{-1}(k_-), \quad (2)$$

where $\Gamma_{5\mu}(P, k)$ is the axial-vector vertex, depending on the total and relative quark momenta P and k , and $S^{-1}(k_\pm)$ the inverse quark propagator with $k_\pm = k \pm P$. The axial-vector vertex has an exact representation via a Bethe-Salpeter equation (BSE)

$$\Gamma_{5\mu}^{ab}(P, k) = - \int_q [S(q_+) \Gamma_{5\mu}(P, q) S(q_-)]^{cd} K_{cd}^{ab}(P, q, k), \quad (3)$$

where K is the Fourier transform of the exact kernel defined through Eq. (1) and $\int_q = \int d^4q / (2\pi)^4$. To proceed we have to define the quark self-energy in the exact form

$$\Sigma(k) = g^2 Z_{1F} C_F \int_q \gamma_\mu S(q) \Gamma_\nu(q, k) D_{\mu\nu}(q - k), \quad (4)$$

with the Casimir $C_F = (N_c^2 - 1)/(2N_c)$, the vertex renormalization factor Z_{1F} , the gluon propagator $D_{\mu\nu}$ and the dressed quark-gluon vertex Γ_ν depending on the incoming and outgoing quark momenta. The self-energy appears in the quark DSE

$$S^{-1}(k) = [S^0(k)]^{-1} + \Sigma(k). \quad (5)$$

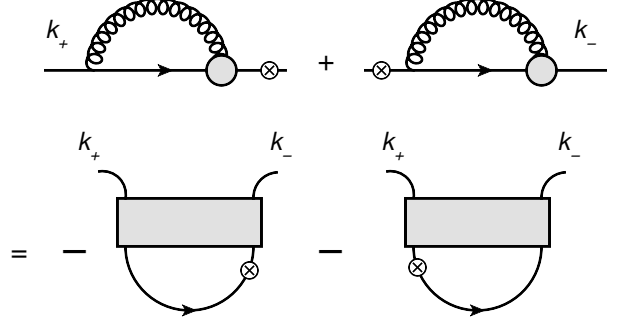


Fig. 1. A graphical representation of the AXWTI shown in Eq. (6). The grey dots are the dressed quark-gluon vertices, grey boxes denote the kernels and the crossed dots represent γ_5 's.

with inverse bare propagator $[S^0(k)]^{-1} = Z_2(-i\not{k} + m)$ including the quark wave function renormalization factor Z_2 . The AXWTI from Eq. (2) can be rewritten in the form

$$[\Sigma(k_+) \gamma_5 + \gamma_5 \Sigma(k_-)]^{ab} = - \int_q [S(q_+) \gamma_5 + \gamma_5 S(q_-)]^{cd} K_{cd}^{ab}(P, q, k), \quad (6)$$

This representation is obtained upon inserting the BSE Eq. (3) in the AXWTI Eq. (2) and using the Dyson-Schwinger equation of the quark Eq. (5). A graphical representation of the resulting expression can be found in Fig. 1.

Before we apply the derivative of Eq. (1) to a given vertex representation we make a short mathematical detour. The space of Euclidean Dirac-matrices with Clifford algebra $\{\gamma_\mu, \gamma_\mu\} = 2\delta_{\mu\nu}$ is spanned by the 16 dimensional basis $T_i = \{\gamma_\mu, \mathbb{1}, \gamma_5, \gamma_5 \gamma_\mu, \sigma_{\mu\nu}\}$ with $\sigma_{\mu\nu} = i/2[\gamma_\mu, \gamma_\nu]$. The elements obey $T_i T_i = \mathbb{1}$ (no summation of indices) and $1/4 \text{tr}[T_i T_j] = \delta_{ij}$. Thus in general, the fully dressed quark propagator and inverse propagator can be represented by

$$S(p) = \sum_{i=1}^{16} T_i \tau_i(p^2), \quad S^{-1}(p) = \sum_{i=1}^{16} T_i \mathcal{A}_i(p^2), \quad (7)$$

where the quark dressings τ and \mathcal{A} depend on the quadratic momentum only. The physical quark propagator has the structure $S(p) = i\not{p} \sigma_V(p^2) + \sigma_S(p^2)$ but in the process of taking the derivative the representation of Eq. (7) is necessary for reasons of completeness. In particular we wish to maintain

$$\begin{aligned} \delta_{ac} \delta_{bd} \delta^{(4)}(p - q) &\stackrel{!}{=} \frac{\delta S^{ab}(p)}{\delta S^{cd}(q)} \\ &= \sum_{i=1}^{16} \frac{\delta \tau_i(q)}{\delta S^{cd}(q)} \frac{\delta}{\delta \tau_i(q)} \sum_{j=1}^{16} T_j^{ab} \tau_j(p) \\ &= \sum_{i=1}^{16} \frac{1}{4} T_i^{dc} T_i^{ab} \delta^{(4)}(p - q), \end{aligned} \quad (8)$$

which, as a completeness relation, can only be valid with the full basis. We used $\delta \tau_i / \delta S^{cd} = 1/4 [T_i^{dc}]^{-1} = 1/4 T_i^{dc}$ and $\delta \tau_j(p) / \delta \tau_i(q) = \delta_{ij} \delta^{(4)}(p - q)$.

The functional derivative onto the quark self-energy Eq. (4) acts on the quark itself, the vertex and the gluon. For simplicity, in the following we disregard derivatives of the gluon propagator. Since the gluon depends on the quark only implicitly via closed loops, contributions from derivatives with respect to the quark only show up in kernels of flavor-singlet mesons. The following discussion is therefore directly applicable only in non-flavor-singlet channels but can be easily generalized to include also the flavor-singlet case.

Although the cutting rule is probably best defined in coordinate space, let us first work in momentum space. On the one hand, this serves illustrational purposes, on the other hand this is necessary for vertex models such as the Ball-Chiu construction [25], which are derived in momentum space. Later on we will demonstrate that the cutting procedure is much simpler in coordinate space and elaborate on a vertex construction (the Munczek vertex [2]) that has a corresponding representation. Cutting the quark propagator, we obtain the *modified* ladder-like contributions (called type *I* in the following)

$$\left. \frac{\delta \Sigma^{ab}}{\delta S^{cd}} \right|_I = \gamma_\mu^{ac} D_{\mu\nu} \Gamma_\mu^{db}. \quad (9)$$

This corresponds to a nonperturbative one-gluon exchange. However, in contrast to the usual ladder kernels one of the quark-gluon vertices is dressed.

For brevity, the kinematic dependences in Eq. (9) are suppressed. On a diagrammatic basis the correct kinematics are easily determined. Yet on a strict mathematical basis the quark is an arbitrary function in the 2PI formalism. Translational invariance cannot be assumed before relaxing the quark to the physical point. Thus in general one has to allow the quark and the self-energy to depend on different ingoing and outgoing momenta. It will be seen below in sections 3.1 and 3.2, that this introduces some complications in the kinematical dependencies of our kernels. These problems are easily overcome when working in coordinate space as will be shown in section 3.3.

Note that with a pure kernel of type *I* the AXWTI from Eq. (6) is fulfilled in the limit $P \rightarrow 0$ only if $\{\Gamma_\mu, \gamma_5\} = 0$. This is because the terms on the right side of the AXWTI assume the form of self-energies for type *I* kernels. The γ_5 has to be moved past the vertices, however. This is trivial for the bare vertex, but non-trivial for more elaborate vertex constructions, pointing towards the necessary appearance of a further type of contributions.

Indeed, a second type of contributions to the interaction kernel contain the variation of the quark-gluon vertex

$$\left. \frac{\delta \Sigma^{ab}}{\delta S^{cd}} \right|_{II} = \int_q [\gamma_\mu S(q)]^{aa'} \frac{\delta \Gamma_\mu^{a'b}(q, p)}{\delta S^{cd}(s)} D_{\mu\nu}(p - q), \quad (10)$$

which is referred to as type *II* contribution. In our notation the variation of the vertex can be decomposed as

$$\frac{\delta \Gamma_\mu^{ab}}{\delta S^{cd}} = \sum_i \frac{1}{4} T_i^{dc} \frac{\delta \Gamma_\mu^{ab}}{\delta \tau_i} \quad (11)$$

where the Dirac indices $\{c, d\}$ are the ones connecting to the incoming quarks. Thus the appearance of certain Dirac structures in the interaction kernel is dictated by whether a corresponding functional variation of the vertex evaluates to zero or not. We will come back to this in the following sections where different vertex representations are considered.

The main observables that we will study to underline our theoretical considerations and test the approach are masses of light mesons in the (pseudo-)scalar and (axial-) vector channels. Their generic Bethe-Salpeter equation for the meson amplitude $\Gamma_M^{(\nu)}$ is given by

$$\begin{aligned} \left[\Gamma_M^{(\nu)} \right]^{ab}(P, k) = \\ - \int_q [S(q_+) \Gamma_M^{(\nu)}(P, q) S(q_-)]^{cd} K_{cd}^{ab}(P, q, k), \end{aligned} \quad (12)$$

with kernel K and the total momentum satisfies $P^2 = -m_M^2$ with m_M the mass of the meson in question. For pseudoscalar mesons, like the pion, the amplitude has the decomposition

$$\begin{aligned} \Gamma_\pi(P, k) = \\ \gamma_5 [E(P, k) + i \not{P} F(P, k) + i \not{K} G(P, k) - [\not{P}, \not{K}] H(P, k)]. \end{aligned} \quad (13)$$

Similar decompositions for the other mesons are given e.g. in Ref. [26]. Furthermore we quote here the Gell-Mann–Oakes–Renner relation (GMOR) [27]

$$f_\pi^2 m_\pi^2 = \langle \bar{\psi} \psi \rangle_\mu m(\mu), \quad (14)$$

which will be a tool to test the chiral properties in our numerical treatment in section 4. Here f_π is the pion decay constant, $\langle \bar{\psi} \psi \rangle_\mu$ the chiral condensate and $m(\mu)$ the running quark mass at renormalization point μ .

3 Constructing the kernel

In the following we show explicitly, how our formalism serves to construct the kernel, once a representation of the quark-gluon vertex in terms of the quark dressing functions is known. For the longitudinal part of the vertex such a representation can be derived (approximately) from its Slavnov-Taylor identity [28]. It reads

$$\begin{aligned} p_3^\mu \Gamma(p_1, p_2) = G(p_3^2) \times \\ \times [H(p_1, p_2) S^{-1}(p_2) - S^{-1}(p_1) H(p_1, p_2)] \end{aligned} \quad (15)$$

in terms of the inverse quark propagator S^{-1} , the ghost dressing function G and a ghost-quark scattering kernel H . The momenta p_1, p_2 correspond to the quark legs of the vertex, whereas $p_3 = p_2 - p_1$ denotes the momentum from the gluon leg. Assuming that $H(p_1, p_2)$ can be approximated by a function $\tilde{H}(p_3^2)$ depending on the gluon momentum only, the STI can be converted into a Ward-Takahashi identity with an extra factor $G \tilde{H}$ on the right

hand side. It is then solved by the Ball-Chiu construction [25] supplemented with the product $G\tilde{H}$

$$\Gamma_\mu^{BC}(p_1, p_2) = G(p_3^2)\tilde{H}(p_3^2) \left[\gamma_\mu \frac{A(p_1^2) + A(p_2^2)}{2} + 2k_\mu \frac{A(p_1^2) - A(p_2^2)}{p_1^2 - p_2^2} + i2p_\mu \frac{B(p_1^2) - B(p_2^2)}{p_1^2 - p_2^2} \right], \quad (16)$$

where $k = (p_1 + p_2)/2$ and vector dressing A and scalar dressing B of the inverse quark propagator

$$S^{-1}(p) = -i\not{p}A(p^2) + \mathbb{1}B(p^2). \quad (17)$$

Within the quark-DSE the functions $G(p_3^2)\tilde{H}(p_3^2)$ can then be combined with the gluon propagator into an effective gluon (cf. Appendix C) and the vertex has an Abelian structure. As a result one has a representation of the vertex in terms of the quark dressing functions. In general, this construction can be supplemented by transverse terms that are not restricted by the STI/WTI and can be either modeled or extracted from explicit solutions of (approximations of) the vertex-DSE. However, for the purpose of this work we restrict ourselves to the Ball-Chiu part of the vertex since it serves nicely to illustrate the merits of our formalism.

In the following we will first treat the first two terms of this vertex ('2BC vertex model'), then deal with the third term in addition ('Ball Chiu vertex model') and finally work with a different solution of the WTI (the 'Munczek vertex model') that is suited to explore the cutting procedure in coordinate space.

3.1 The 2BC Vertex model

Here we consider the first two terms of the parts of the Ball-Chiu vertex Eq. (16) with tensor structures γ_μ and k_μ . Note that these structures correspond to four different structures in the notation of Eq. (7). In order to carry out the cutting for type II kernels along the lines of Eqs. (10) and (11) we therefore write

$$\gamma_\mu A \rightarrow \gamma_\mu A_\mu \quad k_\mu A \rightarrow \sum_\alpha k_\alpha \gamma_\alpha A_\alpha \quad \mu, \alpha \in \{1, 2, 3, 4\}, \quad (18)$$

where no summation over the index μ is performed. The functions obey $A_\mu = \mathcal{A}_\mu/(-ip_\mu)$, i.e. they represent the subset of the \mathcal{A} functions, defined in section 2, that correspond to the γ_μ structures of the Dirac algebra. Via Eq. (16) the A_μ functions are explicitly given in terms of the τ -dressings of the quark in the notation of Eq. (7).

The vertex model Eq. (16) is defined on the physical point of Dirac space and we call the four contributing basis structures γ_i the physical directions in Dirac-space. However, the actual cutting procedure Eq. (1) has to be performed in all directions of Dirac-space, i.e. also in the unphysical ones. In analogy to ordinary functions a functional that is zero at a given point may nevertheless have

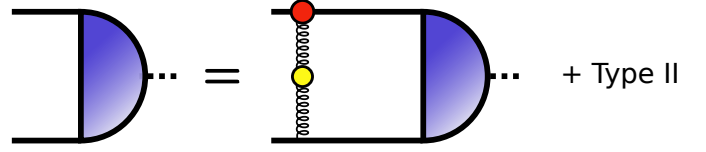


Fig. 2. Bethe-Salpeter equation for mesons including kernel contributions of type I (dressed one-gluon exchanged) and type II (see text).

a non-vanishing functional derivative. Thus the cutting-procedure may very well pick up contributions from the unphysical directions. In order to completely specify a vertex model *and* the corresponding Bethe-Salpeter kernel it is therefore not sufficient to define the model on the physical point, but we need additional information on its behavior in the unphysical directions of Dirac-space.

This is irrelevant for the type I contribution of the kernel obtained from cutting the quark line in the quark-DSE. The resulting expression is afterwards set to the physical point and represents the modified dressed one-gluon exchange shown in Fig. 2. The situation is different, however, for the type II contributions involving the functional derivative of the vertex.

Let us assume for the moment that our 2BC vertex model away from the physical point still has the reduced functional dependence $\Gamma_\mu^{2BC}[A_\mu[\tau_{1...4}]]$ corresponding to $T_{1...4} = \gamma_{1...4}$ and the unphysical directions in Dirac space are identical to zero. This then yields $\delta\Gamma_\mu/\delta\tau_i = 0, \forall i > 4$ such that the external legs of the kernel that will connect to the internal quark lines in the Bethe-Salpeter equation have a restricted tensor structure. In this case, type II contributions to the kernel will appear, but due to their restricted tensor structure they contribute neither to the AXWTI nor to the Bethe-Salpeter equation for pseudoscalar (and axialvector) mesons. This is because of the γ_5 contained on the right hand side of the AXWTI Eq. (6) and in the meson amplitudes of the Bethe-Salpeter equation (12) which lead to zero traces. The explicit form of these type II contributions, relevant for scalar and vector mesons, is discussed in appendix A.

In the AXWTI we thus have to consider only the modified ladder type contributions. Both vertex structures in the first two terms of Eq. (16) anti-commute with γ_5 so that for this particular vertex model the AXWTI Eq. (6) is fulfilled in the limit $P \rightarrow 0$. For the pseudoscalar bound states the modified ladder contributions are all that remains and lead to a massless pion in the chiral limit. This finding will be confirmed by our numerical results in section 4.

Note, however, that the AXWTI is only satisfied in the limit $P \rightarrow 0$. This is because the left and right side of the equation, although similar on the diagrammatic level, need a momentum shift to be absolutely identical. This momentum shift becomes impossible due to the momentum dependence of the first two terms in the vertex (16). In the limit $P \rightarrow 0$, however, the diagrams become equal. For physical pions, a simple vertex model such as the 2BC-vertex cannot be the full story and corrections from un-

physical directions in Dirac space are necessary. In principle, the requirements of chiral symmetry via the AXWTI allow for a systematic procedure to construct such extensions thus completing a given vertex model. We will perform this exercise in the next section. Allowing the functions A_μ to depend on $\tau_{1\dots 16}$, non-vanishing contributions of type *II* in the AXWTI and the pseudoscalar BSEs are generated which can be used to restore the requirements of chiral symmetry also away from the chiral limit. This emphasizes again that a truncation is not uniquely fixed by the vertex model on the physical point.

3.2 The Ball-Chiu vertex model

In addition to the first two terms of the Ball-Chiu type vertex in Eq. (16) we will also consider the third term, which is proportional to the scalar basis element $T_5 = \mathbb{1}$ in Dirac space. Since this term does not anti-commute with γ_5 it cannot fulfill the AXWTI (6) on the level of a pure type *I* ladder kernel. This can, however, be cured by allowing for type *II* contributions to the kernel that couple to the pseudo scalar channel. In order to generate these in a systematic way, we allow the vertex to depend on unphysical components that will be set to zero in the end, but will contribute during the cutting procedure. Therefore, we write the quark generically as

$$\begin{aligned} S(p) &= \left(i \sum_{j=1}^4 \sigma_j p_j \gamma_j \right) + \sigma_S \mathbb{1} + \sigma_5 \gamma_5 \\ S^{-1}(p) &= \left(-i \sum_{j=1}^4 A_j p_j \gamma_j \right) + B \mathbb{1} + C \gamma_5, \end{aligned} \quad (19)$$

with $\sigma_5 = 0$, $C = 0$, $\sigma_j = \sigma_V$ and $A_i = A$ on the physical point. The functions obey $\sigma_j = \tau_j / ip_j, \forall i \leq 4$, $\sigma_S = \tau_5$ and $\sigma_5 = \tau_6$. The reason why we need only six instead of the full sixteen tensor structures in Dirac space is that we will assume a certain functional dependence of the vertex on the quark dressings as in the preceding section. Our vertex, called *ABC-vertex* from now on, reads

$$\Gamma_\mu^{ABC} = \Gamma_\mu^{BC} + i2 \gamma_5 k_\mu \frac{C(k_+^2) - C(k_-^2)}{k_+^2 - k_-^2}. \quad (20)$$

This vertex corresponds to the Ball-Chiu construction for a quark with $C \neq 0 \neq \sigma_5$ as given in Eq. (19). Thus we assume that the vertex does depend only on the quark dressings A , B and C , limiting the possible structures in the type *II* part of the kernel. We furthermore generalize the A function as discussed in Eq. (18). This fully determines the Ball-Chiu type of vertex construction for a quark of the form shown in equation (19).

Now we have laid the basis to explicitly derive the type *II* kernel for the vertex of Eq. (20). The complete set of these kernels is treated in appendix A. It turns out, however, that the application of Eq. (11) generates only one single type *II* kernel that contributes to the AXWTI and the pion BSE after relaxing all dressings to the physical

case. Only the derivative with respect to σ_5 , being accompanied by the γ_5^{dc} structure (see Eq. (11)), will give a non-zero contribution upon tracing with the additional γ_5 as present in the AXWTI (6) and the pion BSA (13).

The relevant piece of Eq. (11) evaluates to

$$\frac{1}{4} \gamma_5^{dc} \frac{\delta \Gamma_\nu^{ab}(l, k)}{\delta \sigma_5(q)} = \frac{1}{4} \frac{i(l+k)_\nu}{l^2 - k^2} \left[\frac{\delta C(l^2)}{\delta \sigma_5(q)} - \frac{\delta C(k^2)}{\delta \sigma_5(q)} \right] \gamma_5^{dc} \gamma_5^{ab}, \quad (21)$$

with

$$\left. \frac{\delta C(l^2)}{\delta \sigma_5(q)} \right|_{phys} = -\frac{1}{4} \frac{1}{\sigma_V^2(l) l^2 + \sigma_S^2(l)} \delta^{(4)}(l - q). \quad (22)$$

The corresponding kernel is then generated by insertion into Eq. (10). The resulting expression is provided in appendix B where we also prove that the AXWTI is fulfilled in the limit of vanishing total momentum.

We would like to emphasize that we make a non-trivial observation here. We nicely see how type *I* and type *II* contributions cancel each other exactly in the AXWTI as is shown explicitly in appendix B. This gives deep insight into the way chiral symmetry is at work in beyond rainbow-ladder truncations in general. In fact it is no coincidence that the generalized Ball-Chiu vertex from equation (20) has the correct behavior. Following the arguments of Ref. [2] a quark-gluon vertex model that transforms under local chiral transformations as an inverse quark should leave chiral symmetry intact in every possible relation derived from the 2PI effective action. A vertex that fulfills the vector WTI, as the BC vertex does, is thus at least a very good candidate for a vertex model. We show here, how these formal arguments are realized explicitly in a Bethe-Salpeter interaction kernel.

There is, however, an additional subtle point here. The momentum space representation of Eq (1), if written as $K = \delta \Sigma(p) / \delta S(l)$, depends only on two momenta, p and l . As a four-point function K should depend on three independent momenta in general $K(P, p, l)$ as is the case for the type *I* interaction $K_I(P, p, l) = \gamma_\mu D_{\mu\nu}(p-l) \Gamma_\nu(l, p_+)$. As argued above, from our cutting procedure this is not plain obvious, instead we complete the kinematic dependence on the diagrammatic level. For the type *II* kernels this is, however, not so simple since these have no representation as Feynman-diagrams. We choose the kinematics such that the potential dangerous singular structure of Eq. (21) stays harmless. This is also detailed in appendix B.

3.3 The Munczek vertex model

Finally we treat a vertex model that has been formulated in coordinate space. We will see, that this choice leads to unique kinematics in the derived kernel and provides for a simple and elegant kernel. The vertex ansatz has been given by Munczek in Ref.[2] and reads in coordinate space:

$$\Gamma_\mu(z; x, y) = iS^{-1}(x, y) \times \int \frac{d^4 q}{(2\pi)^4} \left[e^{iq \cdot (z-y)} - e^{iq \cdot (z-x)} \right] \frac{x_\mu - y_\mu}{q \cdot (x - y)}. \quad (23)$$

Because of the unusual form of this vertex, we repeat a few arguments for this particular choice of vertex, given in [2]:

This vertex transforms under local chiral transformations in the following way:

$$\Gamma_\nu(z; x, y) \rightarrow e^{-i\gamma_5 \tau^l \theta_l(x)} \Gamma_\nu(z; x, y) e^{-i\gamma_5 \tau^l \theta_l(y)} \quad (24)$$

similar to the inverse quark

$$S^{-1}(x, y) \rightarrow e^{-i\gamma_5 \tau^l \theta_l(x)} S^{-1}(x, y) e^{-i\gamma_5 \tau^l \theta_l(y)}. \quad (25)$$

This ensures that the 2PI effective action is invariant under a local chiral transformation which is necessary for the pion to be a Goldstone boson. As in the Ball-Chiu case, this vertex ansatz is free of kinematical singularities and compatible with the vector Ward identity (WTI) in coordinate space:

$$\frac{\partial}{\partial z_\mu} \Gamma_\mu(z; x, y) = i[\delta(y - z) - \delta(x - z)] S^{-1}(x, y). \quad (26)$$

From a technical point of view, the biggest advantage of this vertex, in comparison to the Ball-Chiu one, is the fact, that a representation in coordinate space is available. All the problems of the ambiguous momentum routing, that plagued the cutting procedure for the Ball-Chiu vertex are resolved when applying the cutting procedure to the selfenergy in coordinate space. After the cutting, a transformation back to momentum space is possible and yields a closed expression for the interaction kernel and the vertex. For more technical details we refer to appendix D, presenting here only the results.

With the definition

$$[\hat{S}^{-1}]^\mu := \frac{\partial}{\partial k^\mu} S^{-1}(k) \Big|_{k=k_r+\alpha(k_l-k_r)} \quad (27)$$

the vertex reads

$$\Gamma^\mu(k_l, k_r) = i \int_0^1 [\hat{S}^{-1}]^\mu d\alpha. \quad (28)$$

Here the momentum k_l specifies the incoming left momenta and k_r the outgoing right momenta. They are connected via $k = k_l - k_r$, where k is the outgoing gluon momentum. For the kernel we introduce some shorthand notations:

$$[\hat{\Gamma}_\pi]^\nu := \frac{\partial}{\partial p^\nu} \Gamma_\pi(p; P) \Big|_{p=\tilde{p}+\alpha(\tilde{p}-p)} \quad (29)$$

$$\tilde{\Gamma}_\pi := S(p_+) \Gamma_\pi(P, p) S(p_-), \quad (30)$$

with P the total and p the relative momenta of the two-body bound state and $p_\pm = p \pm P/2$. The resulting kernel

can be written down in a closed form as linear operator. Inserted into the right hand side of the Bethe-Salpeter equation we obtain

$$\begin{aligned} \int [K^{II} \tilde{\Gamma}_\pi]_{ab} = \\ \frac{i}{2} \int_{\tilde{p}} d^4 \tilde{p} \int_0^1 d\alpha [\hat{\Gamma}_\pi]_{b'b}^\nu S_{a'b'}(\tilde{p}_-) \gamma_{aa'}^\mu D^{\mu\nu}(\tilde{p} - p) \\ + [\hat{\Gamma}_\pi]_{aa'}^\nu S_{a'b'}(\tilde{p}_+) \gamma_{b'b}^\mu D^{\mu\nu}(\tilde{p} - p). \end{aligned} \quad (31)$$

This expression is already symmetrized as explained in appendix D. The latin indices represent the Dirac matrix indices. All other additional factors as color etc. are suppressed. It is interesting to see that the selfenergy of the quark, Eq. (4) with Eq. (28) as vertex, and the type II contribution to the BSE, Eq. (31), have the same structure. The only difference, modulo momentum dependence, is the replacement of $[\hat{S}^{-1}]^\mu$ with $[\hat{\Gamma}_\pi]^\mu$. Upon inserting the kernel in Eq. (6) and working out the details it can be seen, that the AXWTI is fulfilled and that the structural similarity plays an important role in doing so.

These findings can be summarized in the following way:

- In order to meet the transformation property of Eq. (24) a vertex model is chosen that depends linearly on S^{-1} with the transformation properties of Eq. (25).
- The additional terms on the rhs of Eq. (6), stemming from the cutting procedure have the same structure as a quark selfenergy *because* the vertex is linear dependent on S^{-1} .
- This additional terms that look like quark selfenergies cancel other terms on the rhs of Eq. (6). This happens in a similar fashion as for the ABC vertex, cf. appendix B.

As shown in Ref. [2] this comes with no surprise: If the vertex transforms in the proper way, the determination of the kernel via cutting of the quark selfenergy yields a interaction that preserves the AXWTI. The linearity on S^{-1} is not necessary, but one has to work much harder to preserve the correct transformation behavior if the vertex is nonlinear in S^{-1} . In the result section we check the GMOR explicitly for this particular choice of vertex.

We make a last comment regarding the similarity between the Munczek vertex and the BC vertex. Despite the unusual form of the vertex in Eq. (31), this vertex has a striking resemblance with the Ball-Chiu vertex in momentum space. This can be seen by carrying out the derivative in Eq. (28) explicitly

$$\begin{aligned} \Gamma^\mu(p_l, p_r) = \int_0^1 d\alpha [2p^\mu A'(p^2) \not{p} \\ + \gamma^\mu A(p^2) + i2p^\mu B'(p^2)]_{p=p_r+\alpha(p_l-p_r)}. \end{aligned} \quad (32)$$

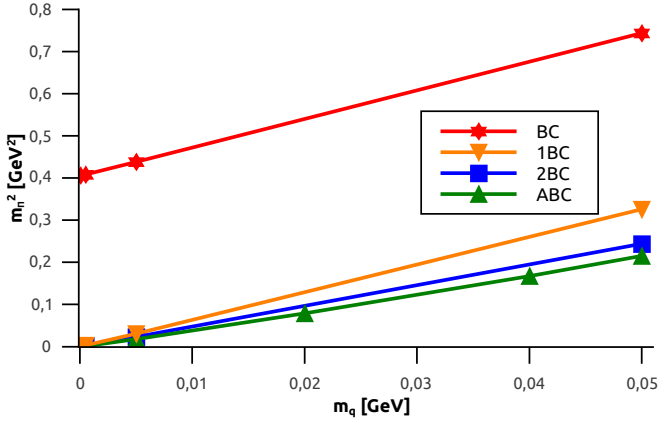


Fig. 3. This figure depicts the dependence of the squared pion mass on the quark mass in the BC-vertex models. Here '1BC' corresponds to a vertex, where only the first term of the Ball-Chiu vertex has been taken into account, '2BC' to the vertex treated in section 3.1, 'BC' is the physical Ball-Chiu vertex dealt with in section 3.2 and 'ABC' is its completion with unphysical directions before cutting, Eq. (20). The quark mass m_q is evaluated at a renormalization point of $\mu = 19$ GeV.

Where the BC vertex has terms that look like finite differences, the Munczek vertex has derivatives smeared by the α integral.

4 Numerical results

For our numerical analysis the quark DSE Eq. (5) was solved for complex momenta following a contour method, described in Ref. [30]. The BSE is solved as an eigenvalue problem with standard numerical methods.

We solved the BSE for the different Ball-Chiu vertex models described in the sections before. Our first main result is shown in Fig. 3. For the 1BC and 2BC vertices, the cutting of the vertex yields no additional contribution to the kernel of the pion, so that the kernel is purely of type *I*. As argued before, the 1BC and 2BC vertex models have only vector contributions that are proportional to γ^μ and thus cause no problems in the AXWTI. As one can see in Fig. 3 the GMOR-relation Eq. (14) is satisfied: the squared pion mass scales linearly with the quark bare mass and goes through the origin.

The results for the full BC vertex are much more intricate. As described above, setting the BC vertex to its physical form before the cutting procedure yields no contribution of type *II* to the pion BSE. Since the scalar parts of the vertex proportional to $\mathbb{1}$ spoil the AXWTI, the resulting equations are not in accord with the requirements of chiral symmetry. This directly translates to a severe violation of the GMOR with a heavy pion of $m_\pi = 637$ MeV in the chiral limit. Adding 'unphysical' directions to the vertex before cutting, however, solves the problem as elaborated in section 3.2. With the resulting ABC-vertex the GMOR-relation is satisfied, and the corresponding curve in Fig. 3 again describes a Goldstone boson.

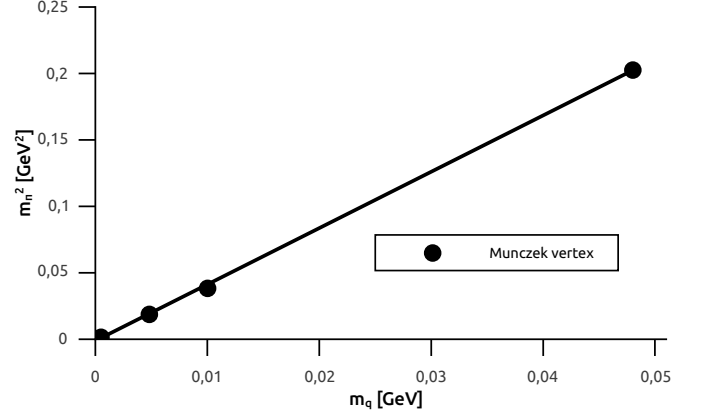


Fig. 4. This figure depicts the dependence of the squared pion mass on the quark mass in the Munczek-vertex model.

Our results for the Munczek vertex model from section 3.3 are displayed in Fig. 4. Again we find that the pion becomes a Goldstone boson in the chiral limit. The main difference as compared to the Ball-Chiu vertex is that we did not have to add terms along unphysical directions. This nicely underlines the main message of Ref. [2]: having the right chiral transformation property already on the level of the vertex representation ensures a massless pion in the chiral limit, provided the kernel is properly constructed. The Munczek vertex preserves all symmetries by design and produces the correct type *II* kernel contributions for the pion automatically.

It is also interesting to discuss the physical implications of such a type of vertex. The Munczek vertex possesses a structure that is proportional to $\mathbb{1}$ and the derivative of the scalar quark dressing function. Such a structure is not present in the chirally symmetric theory and thus represents an important addition to the structure of the vertex that is mainly generated by the dynamical effects of chiral symmetry breaking. This structure is also not present in a usual ladder approximation with a vertex proportional to γ^μ . In the Munczek vertex, this term plays a similar role than the corresponding scalar contribution to the Ball-Chiu vertex. In Ref. [20] this term has been interpreted as being responsible for a dramatic increase in the mass splitting between the scalar and the pseudoscalar ground state and was interpreted as a repulsive spin-orbit force.

We investigated the Munczek vertex for a similar behavior. In order to assess the model dependence of our results we varied the parameters η and Λ in the ansatz for the effective interaction, see Eq. (47). For each value of the parameters we adjusted the bare quark mass to obtain roughly the physical pion mass of $m_\pi = 0.137$ GeV. We then calculated the pion decay constant on the pion mass shell, see [3] for numerical details, and the masses for the pion, the scalar, the vector and the axial-vector ground states. Our results are shown in Fig. 5 and in Tab. 1.

In general, the pion mass serves to fix the input quark mass, whereas the pion decay constant is sensitive to the

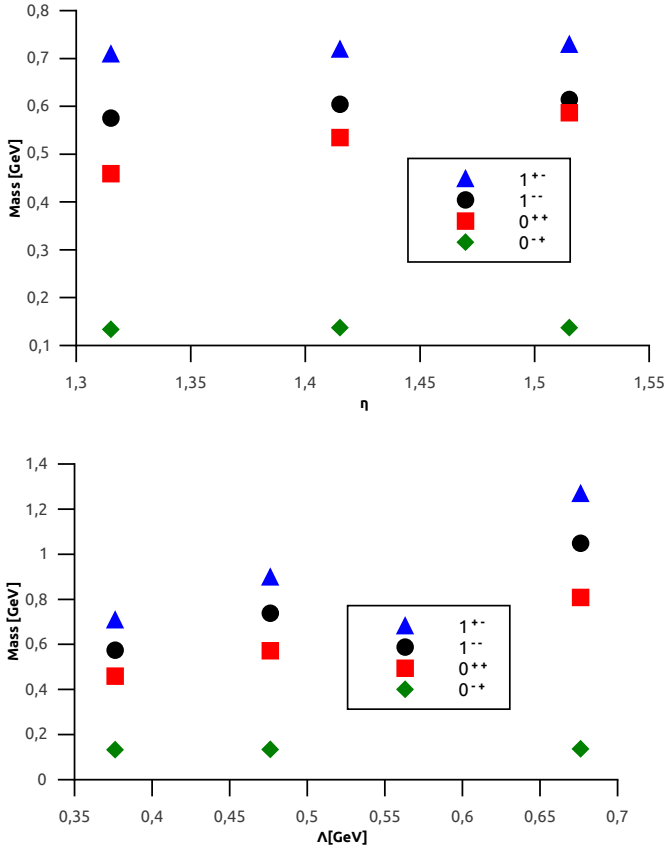


Fig. 5. This figure shows the dependence of the ground state meson masses on the details of the quark-gluon interaction with Munczek vertex. In the upper panel $\Lambda = 0.376$ is held fixed, in the lower panel $\eta = 1.315$.

scale of the interaction. In pure rainbow ladder calculations it has been observed that once the scale is fixed via Λ , there is a whole range of values for η which leave the pion decay constant untouched. It has also been established, that the masses of the scalar and vector meson bound states are almost insensitive to these variations [31]. This is no longer true, when the vertex is non-trivial as can be seen from Fig. 5. Varying the parameters of the interaction one clearly finds a great impact onto the meson mass spectrum. This comes with an increase of the splitting between pseudoscalar and scalar channels as well as vector and axialvector channels. Thus in principle, by variation of the model parameters one could drive the masses of the scalar and axialvector states in a region around and above 1 GeV, where they could be identified with physical states such as the $f_0(1370)$ and the $a_1(1260)$.

However, with the construction at hand this would be stretching the model much too far: as can be seen from Tab. 1 also the pion decay constant increases with increased spin-orbit splitting, clearly indicating that one is no longer working with acceptable model parameters. Indeed, when we compare the rainbow-ladder result (RL) with the improved approximation scheme using the Munczek

| Λ [GeV] | η | f_π [GeV] | | Λ [GeV] | η | f_π [GeV] |
|-----------------|--------|---------------|--|-----------------|--------|---------------|
| 0.376 | 1.315 | 0.094 | | 0.376 | 1.315 | 0.094 |
| 0.376 | 1.415 | 0.103 | | 0.476 | 1.315 | 0.118 |
| 0.376 | 1.515 | 0.109 | | 0.676 | 1.315 | 0.165 |

Table 1. Results for the pseudoscalar decay constants for different parameters in the quark-gluon interaction with Munczek vertex.

| | Λ [GeV] | η | f_π | m_π | m_σ | m_ρ | m_{a1} |
|----|-----------------|--------|---------|---------|------------|----------|----------|
| RL | 0.7 | 1.8 | 0.093 | 0.137 | 0.65 | 0.73 | 0.83 |
| MV | 0.376 | 1.315 | 0.094 | 0.134 | 0.46 | 0.58 | 0.71 |

Table 2. Meson masses and decay constants (in units of GeV) for Rainbow ladder (RL) compared with our results using the Munczek vertex (MV).

vertex (MV) in Tab. 2 with model parameters adjusted such that the pion decay constant comes out right we even observe a decrease of the spin-orbit splitting. Similar results can be obtained with the improved Ball-Chiu vertex (ABC) of section 3.2. We thus find, that a Ward-Identity improved vertex alone is not enough to reproduce the size of the spin-orbit splitting that is suggested from experiment. Note that we do not put much emphasis on the fact, that the mass of the quark-antiquark bound state in the scalar channel using the MV-vertex is even in the right ballpark for the $f_0(500)$. As noted in Ref. [22] there are indeed transverse parts of the vertex that do increase the spin-orbit splitting by a substantial amount thus making the identification with the $f_0(1370)$ more likely. This also ties in with findings of Ref. [32].

5 Conclusions

Following the time-honored concept of taking functional derivatives to obtain an interaction kernel, we extended this technique to vertex models which explicitly depend on the quark propagator and its dressing functions. This enabled us to derive closed expressions for the interaction kernel beyond the rainbow-ladder approximation. Our technique is very general, and in principle applicable to any vertex that is given in terms of quark dressing functions. As an improvement over previous approaches [20, 21, 22] our technique allows to determine not only the masses of the bound states but also their Bethe-Salpeter wave functions. Certainly, these are indispensable when it comes to the calculations of form factors, structure functions, or decay widths of the states in question.

As examples, we applied this technique to two types of vertices, the Ball-Chiu vertex and the Munczek vertex that both respect the constraints due to the vector Ward-Takahashi identity. For the Ball-Chiu vertex we find that we have to amend the vertex by additional parts along unphysical directions in Dirac space. These do not contribute to the Dyson-Schwinger equation for the quark propagator, but generate important additional terms into

the interaction kernel of Bethe-Salpeter equations necessary to respect the axial Ward-Takahashi identity. The resulting pion is then a Goldstone boson in the chiral limit. For the Munczek vertex, such additional contributions are not necessary.

Using the Munczek vertex we performed a calculation of the masses of pseudoscalar, scalar, vector and axialvector mesons and confirm the findings of Ref.[20]: the additional gauge related structure in the vertex is dominated by dynamical effects of chiral symmetry breaking and capable to generate substantial spin-orbit forces. However, these structures alone are not sufficient to generate a physical spectrum of light mesons while keeping the pion properties intact. Additional transverse pieces in the vertex are necessary to improve this situation.

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A Constructing beyond ladder kernels

Here we detail the construction of type *II* kernels in order to provide a self-contained definition that should help the reader who is interested in the numerical implementation. We consider in particular the *ABC* vertex construction from Eq. (20). The quark dressing functions are taken to be the ones from Eq. (19).

The kernels are of the form

$$\frac{\delta \Sigma^{ab}(k)}{\delta S^{cd}(q)} \Big|_{II} = \int_l [\gamma_\mu S(l)]^{aa'} \frac{\delta \Gamma_\mu^{a'b}(l, k)}{\delta S^{cd}(q)} D_{\mu\nu}(l - k), \quad (33)$$

where the vertex from Eq. (20) using the generalisation from Eq. (18) is written as

$$\begin{aligned} \Gamma_\mu^{ABC}(l, k) &= \gamma_\mu \frac{A_\mu(l) + A_\mu(k)}{2} \\ &+ (l+k)_\mu \sum_{\alpha=1}^4 (l+k)_\alpha \gamma_\alpha \frac{1}{2} \frac{A_\alpha(l) - A_\alpha(k)}{l^2 - k^2} \\ &+ i(l+k)_\mu \frac{B(l) - B(k)}{l^2 - k^2} + i\gamma_5 (l+k)_\mu \frac{C(l) - C(k)}{l^2 - k^2}, \end{aligned} \quad (34)$$

which is the analog of the Ball-Chiu construction for the quark shown in Eq. (19). The cutting is now explicitly done as

$$\frac{\delta}{\delta S^{cd}(q)} = \sum_{j=1}^4 \frac{\gamma_j^{dc}}{i 4 q_j} \frac{\delta}{\delta \sigma_j(q)} + \frac{\mathbb{1}^{dc}}{4} \frac{\delta}{\delta \sigma_S(q)} + \frac{\gamma_5^{dc}}{4} \frac{\delta}{\delta \sigma_5(q)} \quad (35)$$

The functional derivatives that occur are of the form

$$\frac{\delta A_i(p)}{\delta \sigma_j(q)} = \frac{\partial A_i}{\partial \sigma_j}(p) \delta^{(4)}(p - q), \quad (36)$$

and similar for the *B* and *C* functions. We need to specify *A*, *B* and *C* in terms of the σ -dressings. The quark and its' inverse defined as in Eq. (19) are related by

$$\begin{aligned} A_i &= \frac{\sigma_i}{\sum_i^4 p_i^2 \sigma_i^2 + \sigma_S^2 - \sigma_5} \rightarrow \frac{\sigma_V}{p^2 \sigma_V^2 + \sigma_S^2} \\ B &= \frac{\sigma_S}{\sum_i^4 p_i^2 \sigma_i^2 + \sigma_S^2 - \sigma_5} \rightarrow \frac{\sigma_S}{p^2 \sigma_V^2 + \sigma_S^2} \\ C &= -\frac{\sigma_5}{\sum_i^4 p_i^2 \sigma_i^2 + \sigma_S^2 - \sigma_5} \rightarrow 0, \end{aligned} \quad (37)$$

where the expressions after ' \rightarrow ' are the ones after $\sigma_{1...4} \rightarrow \sigma_V$ and $\sigma_5 \rightarrow 0$, i.e. the physical ones that are used in all numerical calculations. The 'unphysical' expressions in Eq. (37) are only needed for the cutting procedure during the derivation of the type *II* kernels. The coefficient matrix from Eq. (36) evaluates to

$$\frac{\partial(A_i|B|C)}{\partial \sigma_j} = \frac{1}{\mathcal{N}^2} \begin{bmatrix} D_1 & \Sigma_2^V & \Sigma_3^V & \Sigma_4^V & \Sigma^{VS} & 0 \\ \Sigma_1^V & D_2 & \Sigma_3^V & \Sigma_4^V & \Sigma^{VS} & 0 \\ \Sigma_1^V & \Sigma_2^V & D_3 & \Sigma_4^V & \Sigma^{VS} & 0 \\ \Sigma_1^V & \Sigma_2^V & \Sigma_3^V & D_4 & \Sigma^{VS} & 0 \\ \Sigma_1^{VS} & \Sigma_2^{VS} & \Sigma_3^{VS} & \Sigma_4^{VS} & D_S & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathcal{N} \end{bmatrix}, \quad (38)$$

with

$$\begin{aligned} \mathcal{N} &= p^2 \sigma_V^2 + \sigma_S^2 & D_i &= \sigma_V^2 \left(\left(\sum_{j \neq i} p_j^2 \right) - p_i^2 \right) + \sigma_S^2 \\ D_S &= p^2 \sigma_V^2 - \sigma_S^2 & \Sigma_i^V &= -2\sigma_V^2 p_i^2 \\ \Sigma_i^{VS} &= -2\sigma_V \sigma_S p_i^2 & \Sigma^{VS} &= -2\sigma_V \sigma_S. \end{aligned} \quad (39)$$

Note that momentum p in the equations above will be evaluated as l or k in equation (34). The type *II* kernel for the vertex model from Eq. (20) is now almost fully specified. In addition we adjust the momentum dependence in order to take into account the flow of the total momentum of the bound state through the kernel. This procedure is explained in appendix B for the case of the $\delta C/\delta \sigma_5$ part.

B Massless pion and 3BC vertex

In this appendix we show how the Ball-Chiu vertex (Eq. (17)) can yield a massless pion in the chiral limit via the extended structure of the vertex from Eq. (20). The only type *II* term in the kernel originating from cutting the *ABC* vertex of Eq. (20) and contributing to the AXWTI (6) and the pion BSE (12) evaluates to (see appendix A)

$$\begin{aligned} \frac{\delta \Sigma^{cd}(k)}{\delta S^{ab}(q)} &= -\gamma_5^{ba}/4 \\ &\times \left[[\gamma_\mu S(q) \gamma_5]^{cd} \frac{i(q+k)_\nu}{q^2 - k^2} \frac{D_{\mu\nu}(q-k)}{q^2 \sigma_V^2(q^2) + \sigma_S^2(q^2)} \right. \\ &\left. - \int_l [\gamma_\mu S(l) \gamma_5]^{cd} \frac{i(l+k)_\nu}{l^2 - k^2} \frac{D_{\mu\nu}(l-k)}{k^2 \sigma_V^2(k^2) + \sigma_S^2(k^2)} \delta^{(4)}(q-k) \right]. \end{aligned} \quad (40)$$

We mentioned already in section 2 that the kinematics of the kernels generated is not automatically given by the cutting procedure. The self-energy $\Sigma(k)$ expects the same incoming and outgoing momenta. The kernel that is generated from its derivative should have different momenta k_+ and k_- to match the kinematics needed in the bound state equation (12). If the cutting were carried out in coordinate space, this ambiguity would not arise. In order to arrive at a fully specified kernel one should use Eq. (1) without assuming translational invariance but only relaxing all Green functions to physical ones in the end. We were, however, so far unable to write down the Ball-Chiu construction (17) for a quark with different in- and out-going momenta, or, probably preferable, in coordinate space. In our numerical calculation we thus work with a momentum-shifted version of Eq. (40) which reads

$$K_{II}^{ABC}(P, q, k)_{ab}^{cd} = -\gamma_5^{ba}/4 \quad (41)$$

$$\times \left[[\gamma_\mu S(q) \gamma_5]^{cd} \frac{i(q+k_+)_\nu}{q^2 - k_+^2} \frac{D_{\mu\nu}(q-k)}{q^2 \sigma_V^2(q^2) + \sigma_S^2(q^2)} \right.$$

$$\left. - \int_l [\gamma_\mu S(l) \gamma_5]^{cd} \frac{i(l+k_+)_\nu}{l^2 - k_+^2} \frac{D_{\mu\nu}(l-k) \delta^{(4)}(q-k)}{k_+^2 \sigma_V^2(k_+^2) + \sigma_S^2(k_+^2)} \right].$$

Our reasoning for this expression is twofold. First of all it does respect the fact that the total momentum P that should be part of the kernel, as explained above. Second, the singular terms of the form $1/(q^2 - k^2)$ are potentially dangerous in the integration. This is regularized due to the replacement $k \rightarrow k_+ = k + P/2$, where P is imaginary ($P^2 = -m_\pi^2$). It turns out that with this momentum routing a cancellation between the two types of structures present in Eq. (41) occurs. This cancellation mechanism resembles the vertex structure from Eq. (20), where the same type of denominator occurs. However, since the quotient approaches a form that is reminiscent of a derivative $dC(k^2)/dk^2$ the zero-momentum limit is well defined.

We will now show that the AXWTI, Eq. (6), is fulfilled in the limit of $P \rightarrow 0$. For the case of the type I contribution to the kernel, only the third term of the Ball-Chiu vertex, Eq. (17), is a problem (cf. section 3.1). This is because the γ_5 's on the right hand side of equation (6) have to anti-commute with the vertices to give an additional minus sign to match the left side of the equation (cf. Fig. 1). This works out for the first two components of the BC vertex: $\{\Gamma_\mu^{2BC}, \gamma_5\} = 0$. The third component generates a term with the wrong sign since $[\Gamma_\mu^{3rdBC}, \gamma_5] = 0$. It turns out, however, that the type II contributions to the kernel, Eq. (21), remedy the problem: they equal to twice the same contribution but with opposite sign and therefore effectively switch the sign.

In order to be explicit we will start to check the AXWTI, Eq. (6) for the case of a bare vertex $\Gamma_\mu(q, p) = \gamma_\mu$ in Eq. (9), i.e. the rainbow-ladder case. The essential manip-

ulation in Eq. (6) is

$$- \int_q [S(q_+) \gamma_5]^{cd} K_{cd}^{ab}(P, q, k) \quad (42)$$

$$= - \int_q \gamma_\mu S(q_+) \gamma_5 \gamma_\nu D_{\mu\nu}(k-q)$$

$$= \int_q \gamma_\mu S(q) \gamma_\nu D_{\mu\nu}(k_+ - q) \gamma_5 = \Sigma(k_+) \gamma_5,$$

which then matches a corresponding term on the left side of equation (6). For the case of a generic vertex that fulfills $\{\Gamma_\mu(q, k), \gamma_5\} = 0$ we find

$$- \int_q [S(q_+) \gamma_5]^{cd} K_{cd}^{ab}(P, q, k) \quad (43)$$

$$= - \int_q \gamma_\mu S(q_+) \gamma_5 \Gamma_\nu(q_-, k_-) D_{\mu\nu}(k-q)$$

$$= \int_q \gamma_\mu S(q) \Gamma_\nu(q_- - P/2, k_-) D_{\mu\nu}(k_+ - q) \gamma_5$$

$$\xrightarrow{P \rightarrow 0} \int_q \gamma_\mu S(q) \Gamma_\nu(q, k) D_{\mu\nu}(k-q) \gamma_5 = \Sigma(k) \gamma_5,$$

such that the AXWTI is fulfilled at $P = 0$. For a vertex component, such as the third term of the BC part in Eq. (17) that obeys $[\Gamma_\mu, \gamma_5] = 0$ the contribution has the wrong sign, such that even at $P = 0$ the AXWTI is not fulfilled.

We will see that this problem can be cured by including a contribution of type II. Using the definition $f_\nu(q, k) = (q+k)_\nu/(q^2 - k^2)$, the self-energy for the third BC component on the left hand side of the AXWTI reads

$$\int_q \gamma_\mu S(q) f_\nu(q, k_+) (B(q) - B(k_+)) D_{\mu\nu}(k_+ - q) \gamma_5. \quad (44)$$

The corresponding diagram on the right side of the AXWTI has the opposite sign as already stated above. Therefore we consider now the contribution of the type II kernel from Eq. (40). The corresponding C part of the vertex (20) is zero and does not contribute to the self-energies on the left side of the AXWTI. For simplicity we will use the

function f_ν again and also the function \mathcal{N} from Eq. (39).

$$\begin{aligned}
& - \int_q [S(q_+) \gamma_5]^{cd} K_{cd}^{ab}(P, q, k) = \frac{1}{4} \int_q \text{Tr} [S(q_+) \gamma_5 \gamma_5] \times \\
& \left[\gamma_\mu S(q) \gamma_5 \frac{f_\nu(q, k_+)}{\mathcal{N}(q)} D_{\mu\nu}(q - k) \right. \\
& \left. - \int_l \gamma_\mu S(l) \gamma_5 \frac{f_\nu(l, k_+)}{\mathcal{N}(k_+)} D_{\mu\nu}(l - k) \delta(q - k) \right] \\
& = \int_q \gamma_\mu S(q) f_\nu(q, k_+) \frac{\sigma_S(q_+)}{\mathcal{N}(q)} D_{\mu\nu}(q - k) \gamma_5 \\
& \quad - \int_l \gamma_\mu S(l) f_\mu(l, k_+) \frac{\sigma_S(k_+)}{\mathcal{N}(k_+)} D_{\mu\nu}(l - k) \gamma_5 \\
& = \int_q \gamma_\mu S(q) f_\mu(q, k_+) \left[\frac{\sigma_S(q_+)}{\mathcal{N}(q)} - \frac{\sigma_S(k_+)}{\mathcal{N}(k_+)} \right] D_{\mu\nu}(k - q) \gamma_5 \\
& \xrightarrow{P \rightarrow 0} \int_q \gamma_\mu S(q) f_\mu(q, k) (B(q) - B(k)) D_{\mu\nu}(k - q) \gamma_5. \quad (45)
\end{aligned}$$

Here $\text{Tr}[S] = 4\sigma_S$ was used as well as the definition of the B function in Eq. (37). We see that the last line corresponds to Eq. (44) in the $P \rightarrow 0$ limit. In fact the second contribution on the right side of the AXWTI (6) differs by $S(k_+) \gamma_5 \rightarrow \gamma_5 S(k_-)$ such that in the $P \rightarrow 0$ limit it yields the same contribution. Thus we have the contribution of Eq. (45) twice. Due to the global minus sign that comes from the definition of C in Eq. (37) we subtract the 3BC term from Eq. (44) twice such that the AXWTI is fulfilled in the $P \rightarrow 0$ limit.

C Gluon model

In this work we use a model for the effective gluon propagator $D_{\mu\nu}$ that was given in Ref. [29]. In general the gluon is given in Landau gauge as

$$\tilde{D}_{\mu\nu}(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{Z(k^2)}{k^2}, \quad (46)$$

where the non perturbative content is hidden in the dressing function $Z(k^2)$. In the Dyson-Schwinger equation for the quark propagator this dressing function appears together with the fully dressed non-Abelian quark-gluon vertex. Since all explicit vertices used in this work are constructed along the Abelian Ward-Takahashi identity, the following model for the effective gluon represents a product of the gluon propagator with the remaining non-Abelian dressing effects $G\tilde{H}$ in the vertex, cf. the discussion around

Eq. (17). The model is given by

$$\begin{aligned}
\alpha_{\text{eff}}(k^2) &= \frac{g^2}{4\pi} Z_{1F} Z(k^2) G(k^2) \tilde{H}(k^2) \\
&= \pi \eta^7 \left(\frac{k^2}{\Lambda^2} \right)^2 e^{-\eta^2 \frac{k^2}{\Lambda^2}} \\
&\quad + \frac{2\pi \gamma_m (1 - e^{-k^2/\Lambda_t^2})}{\ln[e^2 - 1 + (1 + k^2/\Lambda_{QCD}^2)^2]}, \quad (47)
\end{aligned}$$

where for the anomalous dimension of the quark we use $\gamma_m = 12/(11N_c - 2N_f) = 12/25$, corresponding to $N_f = 4$ flavors and $N_c = 3$ colors, we fix the QCD scale to $\Lambda_{QCD} = 0.234$ GeV and the scale $\Lambda_t = 1$ GeV is introduced for technical reasons and has no impact on the results. The interaction strength is characterized by an energy scale Λ and the dimensionless parameter η controls the width of the interaction. The precise form of this model does not matter in this work. Ultimately we aim to replace this with a self-consistently calculated gluon propagator, see e.g. Ref. [10], and an appropriate expression for the non-Abelian parts of the vertex.

D Munczek model

To cut the quark selfenergy of the Munczek model, a formulation in position space is necessary. The building blocks read as following:

$$\begin{aligned}
\Gamma^\mu(z; x, y) &= \int_{p', p} e^{-ip'(x-z) - ip(z-y)} \Gamma^\mu(p', p) \\
S(x, y) / D^{\mu\nu}(x, y) &= \int_p e^{-ip(x-y)} S / D^{\mu\nu}(p) \\
A(x, y; P) &= \int_{p_1, p_2} e^{-ip_1 x + ip_2 y} A(p_1, p_2; P). \quad (48)
\end{aligned}$$

The first line represents a vertex in position space, the second a quark or gluon propagator and the third one the Bethe-Salpeter amplitude or wavefunction with total momentum P . We denote two further relations that play a role in the derivations:

$$\int_0^1 d\alpha e^{iq\alpha(x-y)} e^{iq(z-x)} = \frac{e^{iq(z-y)} - e^{iq(z-x)}}{iq \cdot (x-y)}, \quad (49)$$

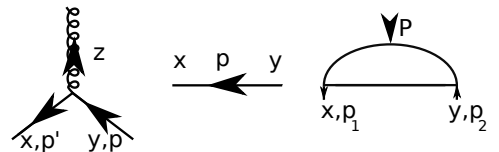


Fig. 6. From left to right: Vertex, Propagator, Amplitude. x, y, z letters denote spacetime positions, p letters denote the corresponding momentum.

and

$$\frac{\delta}{\delta S(l, l')} S^{-1}(x', y) = -S^{-1}(l', y) S^{-1}(x', l). \quad (50)$$

The first one is already used to represent the Munczek vertex model in [2], the second one denotes the functional derivative of an inverse propagator. With all tools at hand the Munczek vertex in momentum space is readily derived from Eq. (23):

$$\Gamma^\mu(p', p) = \frac{\partial}{\partial p_\mu} \int_0^1 S^{-1}(p + \alpha(p' - p)) d\alpha. \quad (51)$$

Taking the functional derivative of the quark selfenergy yields

$$\begin{aligned} \frac{\delta \Sigma(x_1, x_2)}{\delta S(l, l')} &= \int_{y, z} \gamma^\mu S(x_1, y) \frac{\delta \Gamma^\mu(z; y, x_2)}{\delta S(l, l')} D^{\mu\nu}(z, x_1) \\ &= \int_{y, z} \gamma^\mu S(x_1, y) D^{\mu\nu}(z, x_1) \int_q e^{iq(z-y) - iq(z-x_2)} \\ &\quad \times \frac{(x_2 - y)^\mu}{iq \cdot (x_2 - y)} \frac{\delta}{\delta S(l, l')} S^{-1}(y, x_2). \end{aligned}$$

This expression is now traced with the Bethe-Salpeter wave function from Eq.(48) (as demanded by the Bethe-Salpeter Equation in coordinate space) and Eq.(50) is inserted for the derivative of the inverse quark propagator. Additionally the α -trick from Eq.(49) is applied resulting in the following expression

$$\begin{aligned} \frac{\delta \Sigma(x_1, x_2)}{\delta S(l, l')} &= - \int_{y, z} \int_{l, l', q}^1 d\alpha \gamma^\mu S(x_1, y) D^{\mu\nu}(z, x_1) e^{iq\alpha(x_2 - y)} \\ &\quad \times e^{iq(z - x_2)} (x_2 - y)^\mu S^{-1}(l', x_2) \tilde{\Gamma}(l', l; P) S^{-1}(y, l), \end{aligned} \quad (52)$$

where $\tilde{\Gamma}$ is the wave function (see Eq. (30)). Inserting the expressions for the vertex, propagators and amplitudes from Eq.(48) and replacing $(x_2 - y)^\mu$ by appropriate derivatives of momenta in the exponentials of the Fourier modes, one finally arrives at the expression for the type II momentum space contribution:

$$\begin{aligned} [\Gamma \times K_{II}](P, p)|_{ab} &= - \int_q \int_0^1 d\alpha \gamma_{ac}^\mu S_{cd}(q - \frac{1}{2}P) \\ &\quad \times D^{\mu\nu}(q - p) \left[\frac{\partial}{\partial q^\nu} (\Gamma_{db}(q + \alpha(q - p); P)) \right]. \end{aligned} \quad (53)$$

Color factors and renormalization constants are suppressed. In this case Γ denotes the wave function. and instead of the two momenta p_1 and p_2 , we use the relative momentum $p = (p_1 - p_2)/2$ to describe the wave function. We

included the Dirac indices to clarify the structure.

There is an asymmetry in this type II kernel as one can see in the quark momentum. This can lead to an imaginary part of the BSE eigenvalues at least in the form of numerical noise. The source for this is the asymmetry of the quark self-energy that contains only one dressed vertex. If one would start with a symmetrised self-energy

$$\Sigma(p) = \frac{1}{2} \int \gamma^\mu S(q) \Gamma^\nu(q, p) D^{\mu\nu}(p - q) \quad (54)$$

$$+ \frac{1}{2} \int \Gamma^\mu(p, q) S(q) \gamma^\mu D^{\mu\nu}(p - q) \quad (55)$$

this problem disappears and there is second type II contribution containing a quark with momentum $q + \frac{1}{2}P$:

$$\begin{aligned} [\Gamma \times K_{II}](P, p)|_{ab} &= - \int_q \int_0^1 d\alpha \left[\frac{\partial}{\partial q^\nu} (\Gamma_{ac}(q + \alpha(q - p); P)) \right] \\ &\quad \times S_{cd}(q + \frac{1}{2}P) \gamma_{db}^\mu D^{\mu\nu}(q - p). \end{aligned} \quad (56)$$

Both contributions will come with a factor $\frac{1}{2}$.

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